Note on the longitudinal decoherence of a proton/ion bunch

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The question of longitudinal decoherence of the collective motion of a bunch of charged particles in presence of nonlinearities due to a sinusoidal bucket and space charge is addressed in this note.
The method we propose starts from the calculation of decoherence due to nonlinearity alone. Semi-analytical expressions for the evolution of the bunch longitudinal centroid and rms-size (first two moments of the detailed longitudinal particle distribution, on which we assume the space charge forces only depend) are found using a convenient parametrization of the synchrotron detuning with amplitude caused by a sinusoidal bucket. These expressions can then be used as starting point of a recursive numerical procedure, which recalculates the single particle equation of motion with space charge and thus converges to the final profile of longitudinal centroid and rms-size evolution with space charge included after a few iterations.
Macroparticle simulations using the HEADTAIL code are lastly included to benchmark and validate the results obtained with and without space charge.

When a bunch gets longitudinally offset from the bucket center, it begins making synchrotron oscillations around it. The oscillation can be observed with beam position monitors. If all particles have the same synchrotron tune, the centroid motion is expected to be harmonic. However, if the beam contains a spread of tunes, the motion will decohere since the individual synchrotron phases of the particles disperse. As the longitudinal phase space of the beam spreads to an annulus, the observed centroid of the beam will show a decaying oscillation and its rms-size will grow. Strong space charge below transition can inhibit the centroid decoherence and thus keep the oscillations undamped by local compensation of the synchrotron detuning with amplitude. This is what we are going to show in the following.
Different synchrotron tunes are induced by the non-ideal bucket shape, which provides a sinusoidal restoring force. The single particle equation of motion

including space charge can be written as:

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= \dot{z} = -\eta \mathrm{c}\delta \\ \frac{\mathrm{d}\delta}{\mathrm{d}t} &= \mathrm{sgn}(\eta)\frac{\mathrm{e}V_m}{p_0 2\pi R_0}\sin\left(\frac{\omega_{rf} z}{\mathrm{c}}\right) + \mathcal{F}_{sc}(z - <z>, \sigma_z)\end{aligned} \tag{1}$$

where $\delta = \delta p/p_0$, η is the slippage factor defined as $(\gamma_t^{-2} - \gamma^{-2})$, $2\pi R_0$ is the machine circumference, V_m is the holding voltage oscillating at the radio-frequency $\omega_{rf} = h\omega_0$, and $\mathcal{F}_{sc}$ stands for the space charge force normalized to the nominal momentum p_0, which in first approximation depends solely on the particle offset from the bunch centroid $z- <z>$ and on the bunch rms-size σ_z. These equations must then be completed with the proper initial conditions: $(z,\delta)|_{t=0} = (\hat{z}, \hat{\delta})$. Knowing the initial particle distribution in the longitudinal phase space, e.g. let's assume a double Gaussian distribution of a bunch kicked by z_0

$$\rho(\hat{z},\hat{\delta}) = \frac{1}{2\pi\sigma_{z0}\sigma_{\delta 0}}\exp\left[-\frac{1}{2}\left(\frac{(\hat{z}-z_0)^2}{\sigma_{z0}^2} + \frac{\hat{\delta}^2}{\sigma_{\delta 0}^2}\right)\right] \quad , \tag{2}$$

we can also express the centroid and rms-size evolutions through the integrals:

$$\begin{aligned} <z>(t) &= \int_{\Re^2} z(\hat{z},\hat{\delta},<z>,\sigma_z)\rho(\hat{z},\hat{\delta})\mathrm{d}\hat{z}\mathrm{d}\hat{\delta} \\ \sigma_z(t) &= \int_{\Re^2} z^2(\hat{z},\hat{\delta},<z>,\sigma_z)\rho(\hat{z},\hat{\delta})\mathrm{d}\hat{z}\mathrm{d}\hat{\delta} - <z>^2(t)\end{aligned} \tag{3}$$

For simplicity we also assume that the bunch was matched to the bucket prior to offsetting, so that the following equality is satisfied:

$$\sigma_{\delta 0} = \frac{\omega_{s0}\sigma_{z0}}{\omega_0 R_0 |\eta|} \quad . \tag{4}$$

In this equation, $\omega_{s0} = Q_{s0}\omega_0$ represents the linear synchrotron frequency (oscillation frequency at the small amplitudes), which is given by

$$\omega_{s0} = \sqrt{\frac{|\eta|\mathrm{e}V_m h\omega_0}{2\pi R_0 p_0}} \quad ,$$

as can be easily deduced from Eqs. 1.

A first look at the equations (1) and (3) shows that, if space charge is taken into account, the problem consists of a very complicated integro-differential set of equations having $<z>(t)$ and $\sigma_z(t)$ as unknowns with initial conditions $<z>(t=0) = z_0$ and $\sigma_z(t=0) = \sigma_{z0}$. We try to solve it by iterations following the procedure that we describe here below. As first step, we neglect the term $\hat{F}_{sc}(z- <z>, \sigma_z)$ in the second of Eqs. 1. This may allow us to write the solutions of the equations of motion in the closed form:

$$\begin{aligned} z(t) &= -\frac{\eta\mathrm{c}\hat{\delta}}{\omega_s}\sin[(\omega_{s0} - \Delta\omega_s(\hat{z},\hat{\delta}))t] + \hat{z}\cos[(\omega_{s0} - \Delta\omega_s(\hat{z},\hat{\delta}))t] \\ \delta(t) &= \hat{\delta}\cos[(\omega_{s0} - \Delta\omega_s(\hat{z},\hat{\delta}))t] + \frac{\hat{z}}{\eta\mathrm{c}}\sin[(\omega_{s0} - \Delta\omega_s(\hat{z},\hat{\delta}))t]\end{aligned} \tag{5}$$

provided that we can find a parametrization of the synchrotron detuning with amplitude in the form

$$\Delta\omega_s = \omega_0 \Delta Q_s(\hat{z}, \hat{\delta}) = \omega_0 \Delta Q_s(\hat{E}) \ , \tag{6}$$

where

$$\hat{E} = E = \frac{\mathrm{e}V_m \mathrm{c}}{2\pi R_0 \omega_{rf}} \left[1 - \cos\left(\frac{\omega_{rf}\hat{z}}{\mathrm{c}}\right)\right] + \frac{1}{2} p_0 |\eta| \mathrm{c} \hat{\delta}^2$$

represents the particle energy, which is an invariant in the particle motion. In analytical form, this parametrization can be derived from the use of both Eqs. 1 applying a separation of variables and taking into account that the single particle trajectory in longitudinal phase space is uniquely determined by the energy conservation:

$$\Delta\omega_s(\hat{E}) = \frac{|\eta| \mathrm{c} \mathrm{e} V_m}{2 R_0 p_0 \int_0^{\sqrt{2|\eta|\mathrm{c}\hat{E}/p_0}} \frac{\mathrm{d}\dot{z}}{\sin[\omega_{rf} z(\dot{z}, \hat{E})/\mathrm{c}]}} - \omega_{s0}$$

$$z(\dot{z}, \hat{E}) = \frac{\mathrm{c}}{\omega_{rf}} \arccos\left[1 + \frac{\omega_{rf} \pi R_0 p_0 \dot{z}^2}{\mathrm{e}V_m |\eta| \mathrm{c}^2} - \frac{\omega_{rf} 2\pi R_0 \hat{E}}{\mathrm{c}\mathrm{e}V_m}\right]$$

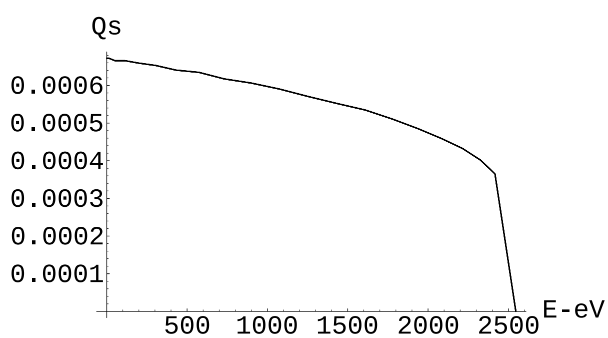

Figure 1: Synchrotron tune Q_s as function of the particle energy (in eV), i.e. of the particle initial amplitudes in longitudinal phase space.

Fig. 1 shows the function $\Delta\omega_s(\hat{E})$ as evaluated numerically from longitudinal tracking of particles having different energies over slightly more than one synchrotron period. We have used the SIS parameters summarized in Table I. The shape of the curve shows that the function stays rather linear over a wide energy range and then steeply drops to zero for energy values that are very close to the bucket boundary. Therefore, we can safely approximate the dependence of the synchrotron tune on the energy with a line having a negative slope k over a large range of energies. If we figure out from the parameters in Table I what the maximum particle energy approximately is in a bunch initially displaced by an amount z_0,

$$E_{max} = \frac{V_m}{2\pi h}\left[1 - \cos\left(\frac{h(z_0 + 2\sigma_{z0})}{R_0}\right)\right] \quad ,$$

we find out that the linear approximation,

$$\Delta\omega_s = k\hat{E} \quad ,$$

can be applied for kick amplitudes up to about $8 \times \sigma_{z0}$. When we use the above equation, we must bear in mind that it only holds if the initial centroid offset inside the sinusoidal bucket does not push the bunch too close to the boundaries of the bucket. However, due to the steep shoulder of the curve plotted in Fig. 1, we can usually be sure that this approximation stays valid even for initial displacements up to many σ's, values far above the kick amplitudes which would be normally imparted in an experiment.

Table 1: SIS parameters used in this study.

variable	symbol	value
Circumference	C	216 m
Relativistic gamma	γ	3.129
Chamber size	a, b	$10 \times \sigma_{x,y}$
Bunch population	N_b	10^{11} p
Rms bunch length	σ_{z0}	2 m
Rms energy spread	$\sigma_{\delta 0}$	5.9×10^{-4}
Slip factor	η	-0.0665
Synchrotron tune	Q_{s0}	6.8×10^{-4}
Betatron tune	$Q_{x,y}$	4.3,3.29
Average beta function	β	8,10.45 m
Rms transv. beam size	$\sigma_{x,y}$	4 mm
Initial kick amplitude	z_0	4 m
Maximum voltage	V_m	32 kV
Harmonic number	h	4

Using now Eqs. 3 with the above parametrization leads us to the following expressions for the bunch centroid and rms-size evolutions in absence of space charge:

$$< z > (t) = -\frac{1}{2\pi\sigma_{z0}\sigma_{\delta 0}}\mathrm{Re}\left[\frac{\mathrm{i}\cdot\exp(\mathrm{i}\omega_s t)}{\sqrt{-\frac{1}{\pi}\left(\frac{1}{2\sigma_{\delta 0}^2} + \mathrm{i}\cdot\frac{p_0|\eta|\mathrm{c}k}{2}t\right)}}\cdot\right.$$

$$\left.\cdot\int_{\Re}\exp\left[-\mathrm{i}\cdot\frac{\mathrm{e}V_m\mathrm{c}k}{2\pi R_0\omega_{rf}}\left(1 - \cos\left(\frac{\omega_{rf}z}{\mathrm{c}}\right)\right)t - \frac{(\hat{z} - z_0)^2}{2\sigma_{z0}^2}\right]\mathrm{d}\hat{z}\right] \tag{7}$$

$$\sigma_z^2(t) + <z>^2(t) = \sigma_{z0}^2 + \frac{z_0^2}{2} - \frac{1}{4\pi\sigma_{z0}\sigma_{\delta 0}} \cdot$$

$$\cdot \mathrm{Re}\left[\int_{\Re} \hat{z}^2 \exp\left[-\frac{(\hat{z}-z_0)^2}{\sigma_{z0}^2} - 2\mathrm{i} \cdot \frac{k e V_m \mathrm{c}}{2\pi R_0 \omega_{rf}} \left(1 - \cos\left(\frac{\omega_{rf}\hat{z}}{\mathrm{c}}\right)\right) t\right] \mathrm{d}\hat{z} \cdot\right.$$

$$\cdot \frac{\mathrm{i} \cdot \exp(2\mathrm{i}\omega_s t)}{\sqrt{-\frac{1}{\pi}\left(\frac{1}{2\sigma_{\delta 0}^2} + \mathrm{i} \cdot p_0 |\eta| \mathrm{c} k t\right)}} + \frac{\eta^2 \mathrm{c}^2}{\omega_s^2} \cdot$$

$$\left. \cdot \int_{\Re} \exp\left[-\frac{(\hat{z}-z_0)^2}{\sigma_{z0}^2} - 2\mathrm{i} \cdot \frac{k e V_m \mathrm{c}}{2\pi R_0 \omega_{rf}} \left(1 - \cos\left(\frac{\omega_{rf}\hat{z}}{\mathrm{c}}\right)\right) t\right] \mathrm{d}\hat{z} \cdot \frac{\exp(2\mathrm{i}\omega_s t)}{\frac{1}{\sigma_{\delta 0}^2} + 4\mathrm{i} \cdot p_0 |\eta| \mathrm{c} k t}\right] \tag{8}$$

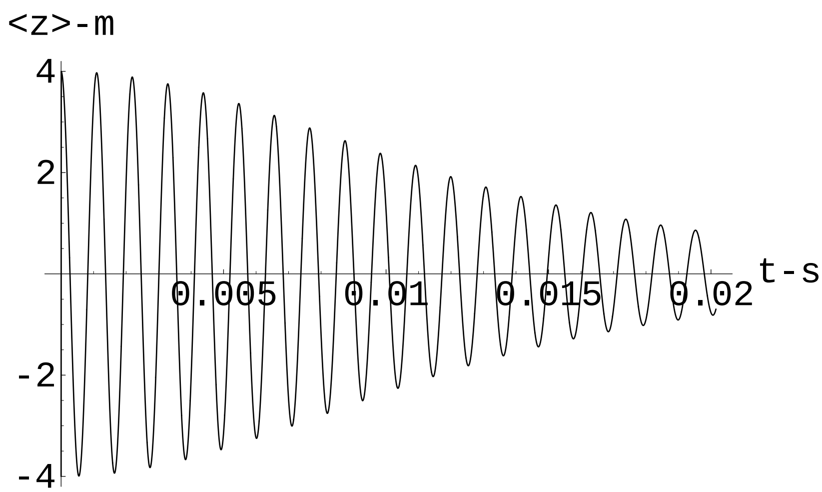

Figure 2: Decoherence of the centroid motion for a longitudinally displaced bunch due to the sinusoidal bucket nonlinearity in absence of space charge effects ($< z >$ in m versus t in s).

Figures 2 and 3 show longitudinal centroid and rms-size evolutions as resulting from the expressions (7) and (8) for parameters in Table I. Due to the sinusoidal bucket the centroid oscillation, which would have survived forever undamped in the case of purely linear restoring force, significantly decoheres after a few synchrotron periods. At the same time the bunch longitudinal rms-size grows and tends to level off at the asymptotic value

$$\sigma_z(t \to \infty) = \sqrt{\sigma_{z0}^2 + \frac{z_0^2}{2}} \ ,$$

as results from Eq. (8) when taking its limit as $t \to \infty$.
Our iterative procedure to evaluate the effect of space charge simply consists in using these evolutions back in Eqs. (1), find new solutions for a set of initial conditions and thus recalculate the integrals (3). After only two iterations the method converges to the evolutions depicted in Figs. 4 and 5. The damping of the centroid oscillation is no longer to be observed, whereas the bunch longitudinal rms-size still increases even if its growth seems to stop at a lower level than without space charge.

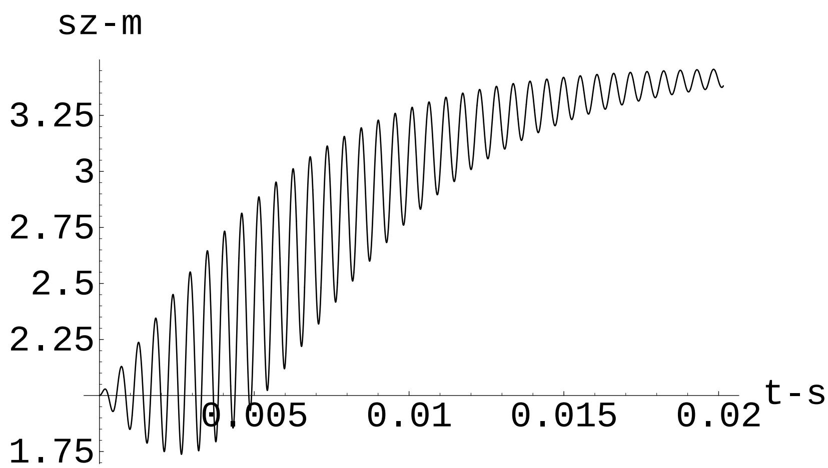

Figure 3: Bunch rms-size evolution for a longitudinally displaced bunch due to the sinusoidal bucket nonlinearity in absence of space charge effects (σ_z in m versus t in s).

To cross-check the validity of the obtained analytical expressions as well as of the iterative procedure which we have used above, we have also carried out macroparticle simulations using the HEADTAIL code with the same parameters of Table I. The results in terms of bunch centroid and rms-size evolution are plotted in Figs. 6 and 7. The agreement with the semi-analytical theory that we have developed is excellent and confirms the capability of space charge to stop the centroid decoherence if nonlinearities are present.

The effect of space charge on longitudinal quadrupole oscillations can also be studied with our method. To excite a pure quadrupole oscillation, we only need to remove the condition (4) and set the initial longitudinal offset of the bunch, z_0, to 0.
In this case, no centroid oscillation will be observed. The σ_z will still evolve according to Eq. (8), where this time it will be $< z > (t) = 0$ and, because of the unmatched situation, σ_{z0}^2 on the first line must be replaced by $\sigma_{z0}^2/2 + \eta^2 c^2 \sigma_{\delta 0}^2/(2\omega_{s0}^2)$. The evolution of the bunch rms-size is depicted in Fig. 8.

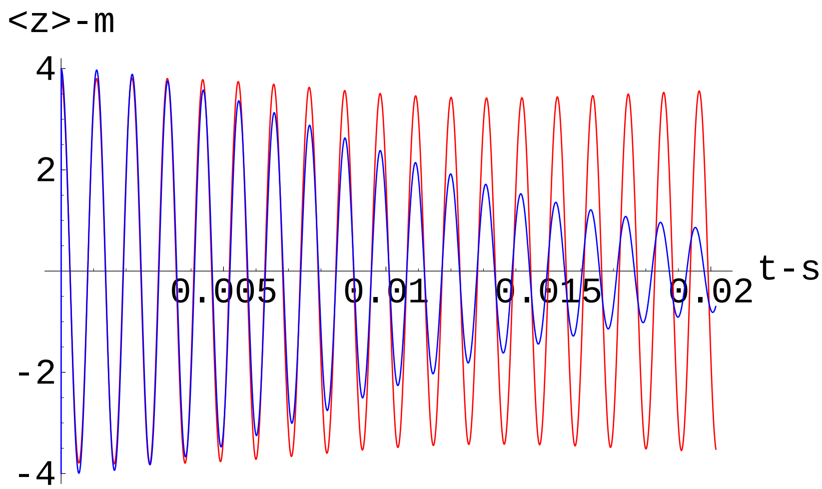

Figure 4: Decoherence of the centroid motion for a longitudinally displaced bunch due to the sinusoidal bucket nonlinearity with (red) and without (blue) space charge effects.

We have chosen to simulate an SIS bunch with a momentum spread which is scaled by a factor 0.8 with respect to the value reported in the Table I (matched value). A smaller momentum spread than the the matched value would cause the bunch to shrink initially and then oscillate around the new $\sigma_z^{(match)} = \sigma_{\delta 0}|\eta|\mathrm{c}/\omega_s = 0.8\sigma_{z0}$ if no longitudinal detuning were included in the analysis. It is clear that, owing to the bucket nonlinearity, decoherence appears in the quadrupole oscillation, too. The asymptotic value of the bunch rms-size, which is eventually reached after the oscillation at twice the synchrotron frequency has fully died out, will be:

$$\sigma_z(t \to \infty) = \sqrt{\frac{\sigma_{z0}^2}{2} + \frac{\eta^2 \mathrm{c}^2 \sigma_{\delta 0}^2}{2\omega_{s0}^2}}$$

Just like in the case of the pure dipole oscillation, space charge causes the quadrupole oscillation to be undamped. Figure 9 shows the persistent oscillation as evaluated with the iteration method.

Figure 10 shows the result of a macroparticle simulation done with the HEADTAIL code.
The fact that the oscillation amplitude reduces when space charge is included can be easily explained by observing that space charge is defocusing below transition, and therefore can lower the net focusing, the strength of which is responsible for the bunch shortening and subsequent oscillation in the unmatched case.

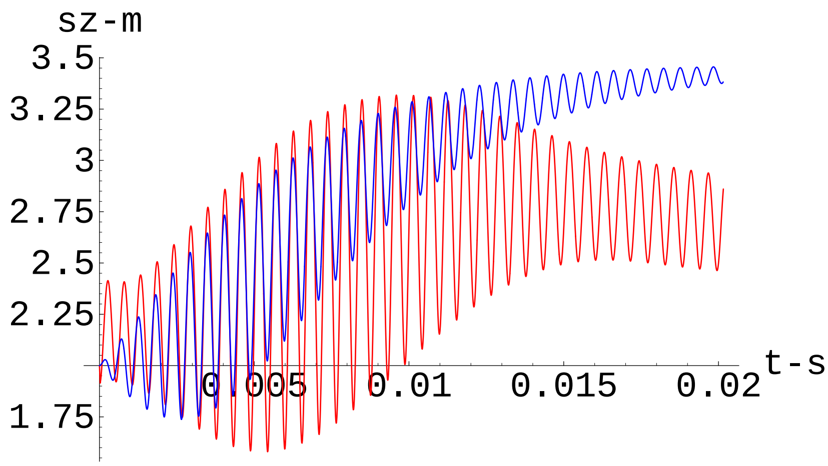

Figure 5: Bunch rms-size evolution for a longitudinally displaced bunch due to the sinusoidal bucket nonlinearity with (red) and without (blue) space charge effects.

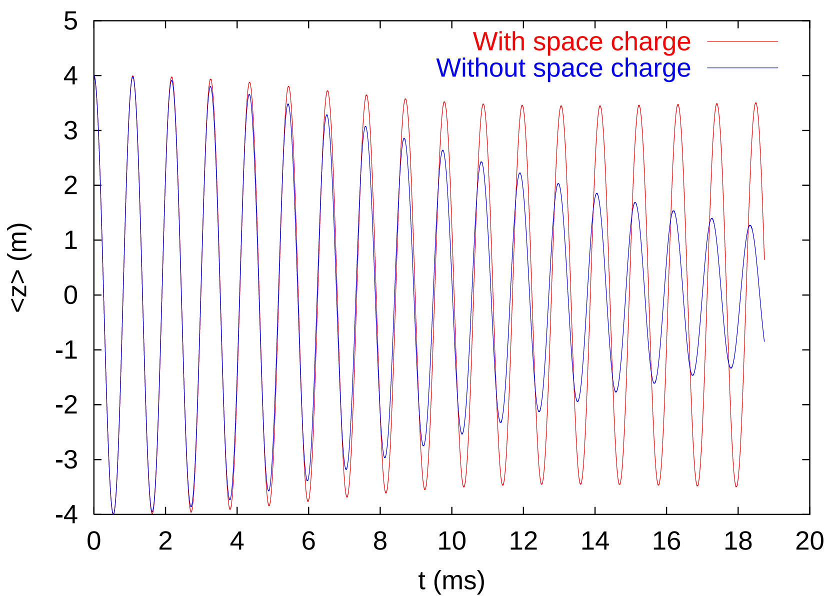

Figure 6: Decoherence of the centroid motion for a longitudinally displaced bunch due to the sinusoidal bucket nonlinearity with (red) and without (blue) space charge effects. This result has been obtained via macroparticle simulation with the HEADTAIL code.

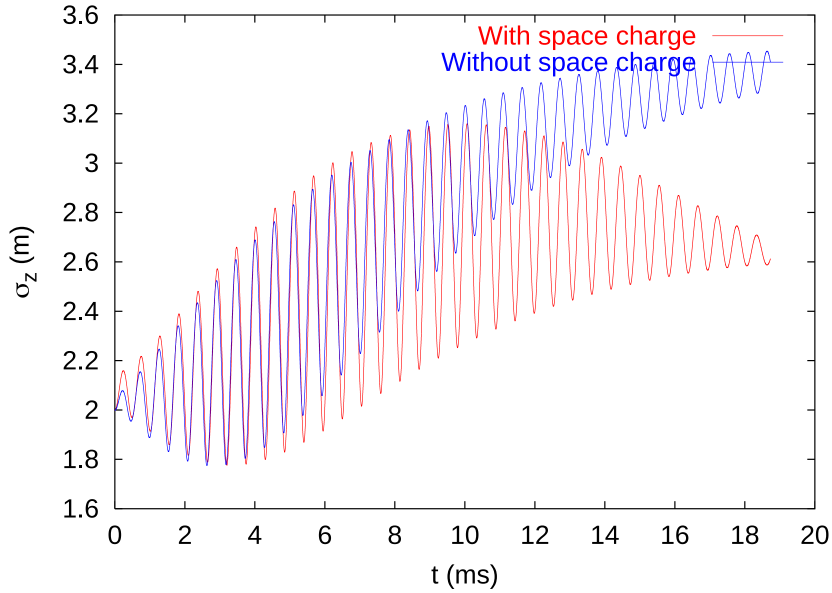

Figure 7: Bunch rms-size evolution for a longitudinally displaced bunch due to the sinusoidal bucket nonlinearity with (red) and without (blue) space charge effects. This result has been obtained via macroparticle simulation with the HEADTAIL code.

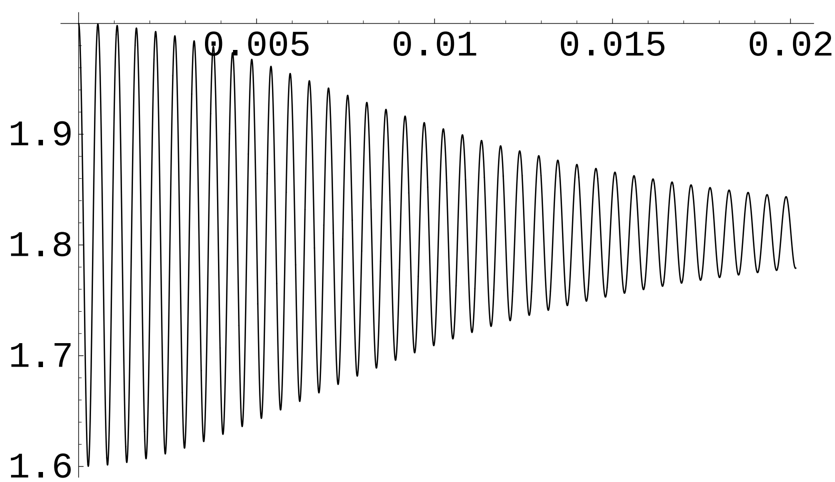

Figure 8: Bunch rms-size evolution for a longitudinally unmatched bunch due to the sinusoidal bucket nonlinearity.

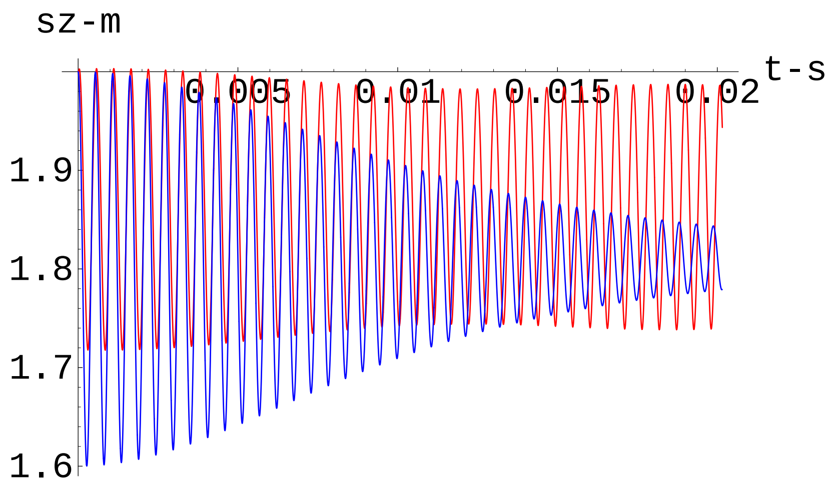

Figure 9: Bunch rms-size evolution for a longitudinally unmatched bunch due to the sinusoidal bucket nonlinearity with (red) and without (blue) space charge effects.

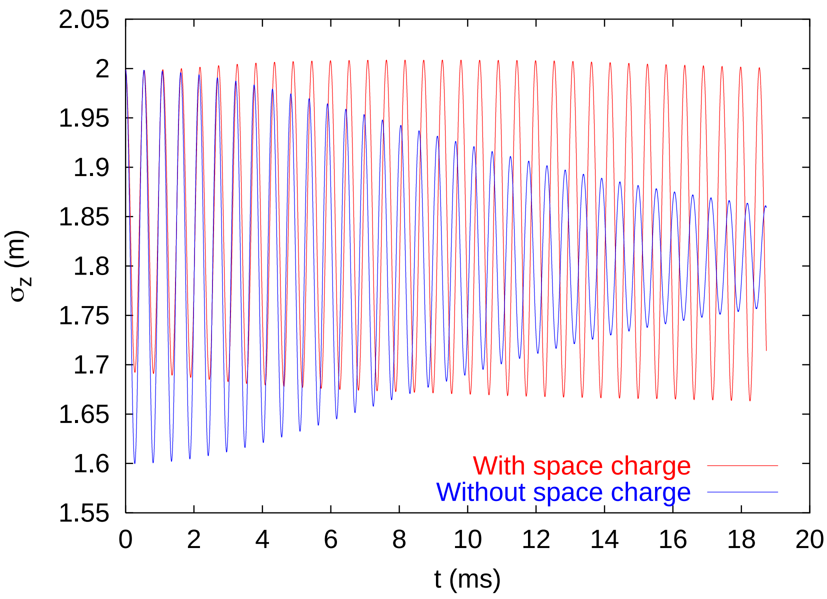

Figure 10: Bunch rms-size evolution for a longitudinally unmatched bunch due to the sinusoidal bucket nonlinearity with (red) and without (blue) space charge effects. This result has been obtained via macroparticle simulation with the HEADTAIL code.